Title
Investigating Adults’ Strategy Use During Proportional Comparison

Journal
Proceedings of the Annual Meeting of the Cognitive Science Society, 44(44)

Authors
Hurst, Michelle
Piantadosi, Steven

Publication Date
2022

Peer reviewed

Investigating Adults' Strategy Use During Proportional Comparison

Michelle A. Hurst (hurstm@uchicago.edu)
Department of Psychology, 5848 S University Ave
Chicago, IL 60637 USA

Steven T. Piantadosi (stp@berkeley.edu)
Department of Psychology, 2121 Berkeley Way
Berkeley, CA 94704 USA

Abstract

Adults show numerical interference during discrete proportional reasoning. Although children's similar errors are attributed to incorrect counting strategies, it is unlikely that adults use a counting strategy. We investigate two behavioral phenomena of proportional reasoning, numerical interference errors and holistic ratio-dependent responding, and use a Bayesian model-based approach to test whether these behavioral patterns can be explained by adults' differential use of numerator comparison versus proportion comparison strategies. We find evidence of numerator interference and holistic ratio dependent responding for both discrete (i.e., individual dots) and continuous (i.e., undivided pie charts) proportions, but numerical interference is stronger for discrete stimuli. Importantly, adults' continuous proportion comparisons were best captured by a proportion strategy, whereas discrete proportion comparisons showed a mixed pattern, with a slight preference for a numerator strategy. These findings provide insight into the mechanisms underlying proportional reasoning and provide a novel model-based approach for investigating strategy use.

Keywords: Proportion; Bayesian Mixture Model; Numerical Interference; Whole Number Bias

Introduction

Reasoning about proportional information is ubiquitous in everyday life, including medicine (e.g., drug dosages), finance (e.g., interest rates), cooking (e.g., scaling ingredients), and many other life decisions (e.g., teacher-student ratio in preschools). Moreover, modern theories of cognitive development rely on infants' abilities to make probabilistic inferences based on proportional information as a central learning mechanism (e.g., Denison & Xu, 2012). For example, infants can make probabilistic inferences between samples and populations (e.g., Denison et al., 2013; Denison & Xu, 2010; Xu & Denison, 2009; Xu & Garcia, 2008) and appreciate the role of random sampling in probabilistic inference (e.g., Kushnir et al., 2010; Xu & Garcia, 2008).

Despite the ubiquity of proportional information and infants' ability to readily use it (although see Placì et al., 2020; Téglás et al., 2015), other research finds that toddlers, older children, and adults often show difficulty with probabilistic and proportional information (e.g., Boyer et al., 2008; Bryant & Nunes, 2012; Fazio et al., 2016; Girotto et al., 2016; Hurst et al., 2021; Schneider & Siegler, 2010; Tversky & Kahneman, 1974). One contributing factor to people's difficulty with proportion is often referred to as the whole number bias (e.g., Ni & Zhou, 2005).

The whole number bias is generally defined as a tendency to treat a fraction or proportion as two separate whole numbers, rather than considering the holistic value. This results in systematic errors, such as thinking that 2/3 is less than 4/9 because 2 is less than 4, calculating that 1/4 + 2/3 = 3/7 by adding the numerators and denominators separately, or placing 4/6 higher on the number line than 2/3 (Bonato et al., 2007; Braithwaite & Siegler, 2018; Ni & Zhou, 2005; Schneider & Siegler, 2010). Although much of the research on the whole number bias has focused on symbolic fractions and how to improve fraction instruction, a similar over attention to numerical components is evident in children's and adults' processing of non-symbolic proportion. For example, when asked to compare or match non-symbolic visual proportions presented as sets of dots or divided shapes, the numerical magnitude of the "numerator" (i.e., the referent or most salient subset) interferes with people's processing of the holistic proportion (e.g., Boyer et al., 2008; Fabbri et al., 2012; Hurst et al., 2021; Hurst & Cordes, 2018). This interference from the discrete components in non-symbolic proportional reasoning is often referred to as "numerical interference". Throughout, we will contrast numerical interference with processing of the holistic proportion (i.e., the proportion magnitude value without interference from the components that make up the proportion).

With children, this numerical interference is often attributed to an over reliance on a counting strategy that is available only when the underlying amounts the proportion is based on are countable. When 6-year-old children were asked to match proportions depicted as a mixture of juice and water, they incorrectly matched on the number of juice units when the visual proportions (i.e., the sample target and the response options) were presented with discrete countable units (Boyer et al., 2008). However, when all visual proportions were presented without countable units (i.e., continuously) or when only one had countable units, children more often matched on proportion instead, presumably because a count-and-match strategy was no longer available (Boyer et al., 2008). As further evidence that numerical interference errors are caused by strategy selection, they are easily mitigated within a single session. Specifically, when children's attention is directed toward the holistic proportion (e.g., through practice with continuous proportional amounts or by using proportion category labels) immediately before making

In J. Culbertson, A. Perfors, H. Rabagliati & V. Ramenzoni (Eds.), *Proceedings of the 44th Annual Conference of the Cognitive Science Society*.

judgements of discrete proportion, they less often show whole number interference (e.g., Boyer & Levine, 2015; Hurst & Cordes, 2019).

Importantly, adults, like children, also show some evidence of numerical interference in non-symbolic proportional reasoning. Specifically, when asked to compare two proportions or match a target proportion with one of two options, the absolute numerical features of the displays interfered, resulting in worse performance when a numerical response option was in competition with the proportional response (Fabbri et al., 2012; Hurst et al., 2021). However, given that adults are unlikely to use a strict counting strategy, the mechanism of this numerical interference in adults is unclear. Furthermore, if the interference is not due to counting then it may also arise for judgements of continuous proportion. In other words, the relevant and salient subset might interfere with adults' proportional reasoning, regardless of whether the information is numerical. If this is the case, then we would expect absolute interference from continuous area as well as discrete number. If it is caused specifically by numerical interference, however, then adults should be adept at reasoning proportionally with continuous amounts regardless of the relative size of the subsets.

In either case, adults' numerical interference during proportional reasoning is unlikely to be explained by a count-and-compare strategy and may instead be an estimate-and-compare strategy. For example, when judging which of two gumball machines are most likely to result in a red gumball, adults might estimate the number of red gumballs in each machine and compare those magnitudes, without considering the number of other gumballs available. Similarly, when comparing proportion based on continuous amounts (e.g., area), the salient magnitude can be approximately compared without considering the other components. If people are using this strategy, then their performance should depend on the absolute magnitudes of the numerator component and how difficult they are to estimate and compare.

Substantial research has investigated people's ability to compare absolute magnitudes and typically find that performance is dependent on Weber's law and shows ratio dependent discrimination, meaning that two magnitudes are easier to compare as the ratio between them increases (e.g., Dehaene, 1992). To account for these behavioral findings, our mental representation of numerical magnitudes, referred to as the Approximate Number System (ANS), is typically modeled as a Gaussian distribution centered at the numerical value and with some noise making the representation approximate (e.g., Cantlon et al., 2009; Dehaene, 1992, 2011; Halberda & Feigenson, 2008). The exact nature of the ANS is debated, but one common theory posits that our numerical representations are represented with linearly increasing noise as a function of the numerical value (e.g., Gallistel & Gelman, 1992; Meck & Church, 1983; although see Dehaene (2003) for an alternative account). This proposal has motivated substantial research generating psychophysical models of numerical comparison, which we will use as a model of numerator comparison.

As with discrete number, comparing continuous extent (i.e., area) also aligns with Weber's law and shows ratio dependent discrimination (e.g., Brannon et al., 2006; Odic et al., 2013). Given these similarities, some have proposed that number and area, as well as other magnitudes, are represented within a common magnitude system (e.g., Walsh, 2003). Although this is strongly debated, with many arguing that discrete number is unique and distinct from continuous magnitude (e.g., Cordes & Brannon, 2008; Odic, 2018), we model area-based numerator discrimination using the same psychophysical model as for discrete magnitudes.

Finally, despite the interference from absolute components during proportional reasoning, children and adults do show evidence of being able to attend to the holistic magnitudes of proportional information. Similarly to the behavior described for absolute magnitude comparisons, people's performance comparing proportions tends to improve as the proportions being compared get further apart (e.g., Kalra et al., 2020; Park et al., 2020).

The Current Study

In the current study, we have two primary research goals. First, we aim to replicate and extend the prior work describing behavioral patterns in proportional reasoning by investigating whether numerical interference and ratio dependent responding (i.e., increasing difficulty with increasing ratio) are evident in adults' proportional reasoning and whether they are features of proportional reasoning in general or are dependent on the underlying quantities the proportion is based upon (i.e., discrete number versus continuous area). Second, and most importantly, we develop a model-based tool for quantifying and comparing people's reliance on specified strategies during proportional reasoning. This approach allows us to investigate process-based strategies in a novel way that is theoretically motivated by the often-reported behavioral patterns found in children's and adults' proportional reasoning. Here, we compare three strategies: a holistic proportion comparison strategy, a comparison of the numerators, and random guessing.

Method

Participants

One hundred and nine adults (M_{age} = 26 years, Range: 18 to 63 years; 76 women, 33 men) are included in the analyses. Adults were recruited from participant databases that included university students and community members and participated entirely online. Adults received course credit or $5. An additional 8 complete datapoints were excluded because they were repeat participants. Prior to completing this task, participants completed a separate experiment investigating their use of quantitative information when making social evaluations of others (Hurst et al., 2020) and the sample size was chosen to provide adequate power for this other study.

Stimuli

Adults completed 80 trials across two blocks, presented randomly within a block. Each block contained 40 unique trials, 10 from each of four ratio bins to ensure variability in closeness between the two proportions (larger proportion of red/smaller proportion of red): 1.06, 1.25, 1.5, and 2 (ratio values were chosen based on prior work with absolute quantities; e.g., Odic, 2018). The blocks differed in the format of the stimuli: discrete proportion or continuous proportion (see Figure 1).

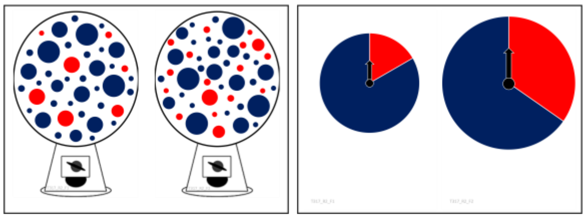

Figure 1: Example trial from the discrete trials (left) and the continuous trials (right).

Discrete stimuli were presented as red and blue dots intermixed within a dispenser. The number of dots of a single color ranged from 6 to 41 and the total number of dots ranged from 14 to 50. The sizes of the dots within a stimulus varied so that the red:blue ratio in terms of surface area did not correspond to the red:blue ratio in terms of number. Fraction comparisons were selected so that the stimulus with the higher number of red items also had the higher proportion of red items on half the trials.

Continuous stimuli were presented as circular spinners with a red portion and a blue portion, and a black arrow extending upward from the center of the circle along a red-blue boundary. The same fractions used in the discrete trials were used in the continuous trials. On each trial, the sizes of the two spinners differed so that the stimulus with the higher red area also had the higher proportion of red on half the trials.

Procedure

Adults completed both the discrete and continuous blocks and were randomly assigned to complete the discrete block first (n = 54) or the continuous block first (n = 55). The procedure within each block was identical and all that differed was the format of the stimuli. Written instructions were provided on the screen prior to each block. The instructions introduced participants to the color machines (dispensers in the discrete block and spinners in the continuous block) and instructed participants to select the color machine that had a higher probability of resulting in red. Participants responded by pressing the right or left arrow key for the right or left stimulus, respectively, and were told to respond as quickly as possible. Stimuli remained visible until a response was selected. We did not restrict the type of device participants used to participate in the study.

Analytical Approach

As our primary approach, we analyzed the experimental data using a Bayesian model over strategies which inferred the probability that each strategy was used in each condition. This approach uses mixture modeling to estimate the parameters within each model and the probabilities given to each strategy. For each observation, the model computes the predicted accuracy based on each of the three defined strategies: numerator comparison, holistic proportion comparison, and guessing. These predictions are weighted and mixed together to form an aggregate prediction. We infer the mixture weights using a No-U-Turn sampler (Hoffman & Gelman, 2014) and these are the primary measures we report. The weights can be viewed as quantifying the amount of support for each process. For example, if the model assigned weights of .60 to the numerator comparison strategy, .30 to the proportion comparison strategy, and .10 to the guessing strategy, this would suggest that the numerator comparison strategy is twice as likely as the proportion comparison strategy. The weights for each model were defined using a softmax function, with a normal(0,3) prior on the parameters.

Models were computed using rstan version 2.21.1 (Stan Development Team, 2020) with four chains each with 10000 iterations. Convergence was determined using Rhat, which were equal to 1 for all parameters. The model weights were estimated separately for the discrete and continuous stimuli. Mean estimates and the 50% equal tailed interval around the mean (i.e., 25% to 75% quantiles) were computed using the posterior samples as measures of central tendency and range.

We next describe how we defined each of the three processes used to determine strategy use. Both the numerator comparison strategy and the proportion comparison strategy rely on the same basic process, which is based on an approximate magnitude system using Weber noise, but differ in the magnitudes being compared and how they relate to accuracy.

Numerator Comparison Strategy The numerator comparison strategy assumes people compare only the numerators (i.e., amount of red) and select the option with more red, ignoring the amount of blue or the total amount. On trials where the correct response has both the higher proportion and the higher numerator, this strategy will predict selection of the correct response. However, on trials where the correct response has a smaller numerator but the higher proportion, this strategy will predict the incorrect response. Thus, this strategy can be modeled as a simple comparison of the number of red dots (discrete stimuli) or the area of red (continuous stimuli), dependent on weber's law. The psychophysics of these judgements has been well studied (e.g., Halberda & Feigenson, 2008; Piantadosi, 2016) and is often formalized as in Equation 1, where Φ is a cumulative normal distribution function, n_1 and n_2 are the numerator magnitudes being compared, and w_n is the weber ratio, determining the precision with which the numerator values are discriminated.

$$P(larger\ numerator) = \Phi\left[\frac{|n_1 - n_2|}{w_n\sqrt{n_1^2 + n_2^2}}\right] \quad \text{(Eq 1)}$$

The numerators being compared (n_1 and n_2) are set by the experiment design for each trial. The weber ratio, w_n, is estimated based on participants' accuracy, with an exponential prior, exp(1). Accuracy on each trial was predicted based on the probability of selecting the larger numerator (as in Equation 1) for trials where the larger proportion had the larger numerator and 1 – the probability of selecting the larger numerator on trials where the larger proportion had the smaller numerator.

Proportion Comparison Strategy The ratio comparison strategy assumes participants compare the holistic proportions, integrating information about the amount of red and the amount of blue. In line with this strategy, both adults' and children's performance comparing symbolic fractions, decimals, and non-symbolic proportion is dependent on the closeness of the two proportion magnitudes (e.g., DeWolf et al., 2014; Hurst & Cordes, 2016; Kalra et al., 2020; Park et al., 2020). We modeled this strategy as an approximate Weber-based comparison on the proportions, using the same approach as in the numerator comparisons, where Φ is a cumulative normal distribution function, r_1 and r_2 are the proportion magnitudes being compared, and w_p is the weber ratio, determining the precision with which the proportion values are discriminated (see Equation 2)

$$P(correct) = \Phi\left[\frac{|r_1 - r_2|}{w_p\sqrt{r_1^2 + r_2^2}}\right] \quad \text{(Eq 2)}$$

The proportions being compared (r_1 and r_2) are set by the experiment design for each trial. The weber ratio, w_p, is estimated based on participants' accuracy, with an exponential prior, exp(1). Here, unlike the numerator comparison, this model predicts the probability of selecting the larger proportion for all trials.

Guessing As a final model, we also included guessing, which is simply a 50% chance of responding correctly on each trial.

Openness and Transparency

All materials, data, and analysis code are available on the Open Science Framework (OSF; https://osf.io/bescn/). The sample size, basic design, and an analysis plan was preregistered at AsPredicted.org (#41509). The current paper deviates from the pre-registration in two ways. First, we pre-registered a sample of 108, but due to issues with randomization and drop out we ended up with one extra participant. Second, the preregistered analyses focused on a traditional frequentist approach to comparing performance across different trial types. However, as described in the Analytical Approach section, the focus of the current paper is to take a model-based approach to data analysis, which was not included in the preregistration. We include only a brief summary of adults' performance using traditional approaches to investigating numerical interference and ratio dependent responding to motivate the novel model-based approach.

Results

The two competing processes we investigate here are motivated by two behavioral patterns found in prior work: (a) dependence on the congruency between absolute numerator information and the proportional information and (b) dependence on the ratio of the proportions being compared. However, how these behaviors differ across proportional reasoning with continuous and discrete quantities is an open question. Thus, we first test whether these behavioral patterns are evident in the current data and extend this prior work to investigate whether they differ for proportional reasoning based on discrete versus continuous quantities in adults (see Table 1 for means and standard deviations).

Table 1: Mean (standard deviation) proportion correct

		Continuous Trials	Discrete Trials
	Overall Performance	.82 (.14)	62 (.13)
Numerator Interferenc	Numerator Consistent	.90 (14)	.78 (.17)
	Numerator Inconsistent	.74 (.17)	.47 (.22)
Ratio Dependence	Ratio 1.06	.71 (.17)	.53 (.14)
	Ratio 1.25	.77 (.15)	.62 (.17)
	Ratio 1.5	.88 (.18)	.60 (.21)
	Ratio 2.0	.91 (.15)	.73 (.19)

Numerical Interference

Numerical interference is typically demonstrated by comparing performance on trials where the numerator is consistent with the proportion (i.e., the larger proportion also has the larger numerator) to trials where the numerator is inconsistent with the proportion (i.e., the larger proportion has the smaller numerator). We do see evidence of numerical interference from the numerator component on both continuous, $t(216) = 7.23$, $p < .001$, $d = 0.95$, and discrete trials, $t(216) = 11.62$, $p < .001$, $d = 1.07$. However, there is a significant interaction between stimulus type and numerator consistency, $F(1, 108) = 24.48$, $p < .001$, partial $\eta^2 = 0.19$, revealing larger numerical interference for discrete trials than continuous trials (see Figure 2).

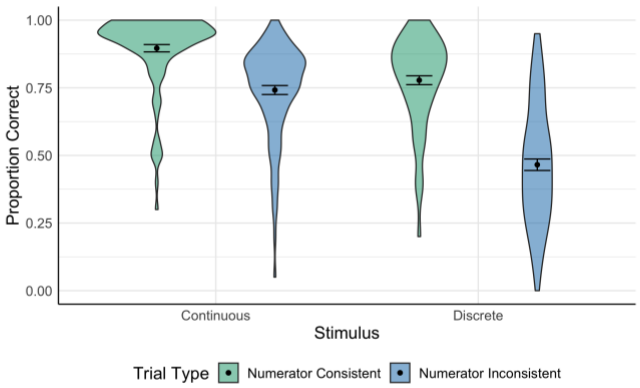

Figure 2: Proportion correct on continuous and discrete trials as a function of the consistency between the numerator magnitude and the proportion magnitude.

Ratio Dependence

Linear mixed effects models, with fixed effects of ratio category, stimulus format, and the interaction, as well as a random intercept for participants (Gelman & Hill, 2007), reveal significant effects of ratio category for both continuous, B = 0.21, $p < .001$, and discrete trials, B = 0.19, $p < .001$, but counter to our hypothesis, the interaction was not significant, B = 0.03, $p = .34$ (Figure 3).

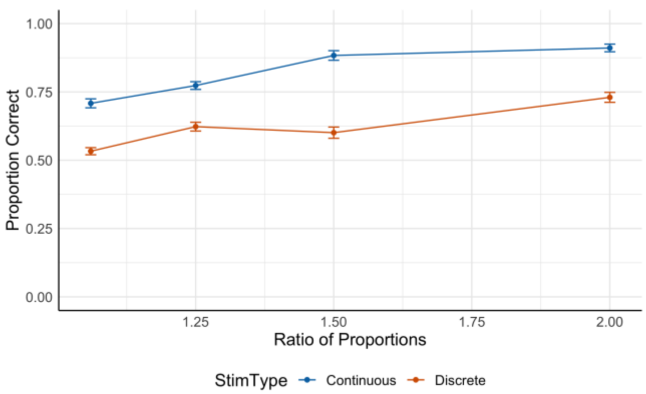

Figure 3: Proportion correct on continuous and discrete trials as a function of the ratio category between proportions being compared (ratio = larger proportion/smaller proportion, grouped into four categories as described in the Method).

Bayesian Strategy Discovery

Thus, adults' behavior shows evidence of numerical interference and ratio dependent responding for both discrete and continuous proportional reasoning, but with larger numerical interference for discrete proportional reasoning. Here we use a novel Bayesian model-based approach (as described in the Analytical Approach section) to compare two competing processes that may give rise to these different patterns of behavior. We report the mean mixture weights and corresponding 50% equal tailed interval for each of three possible strategy models with both formats (Figure 4).

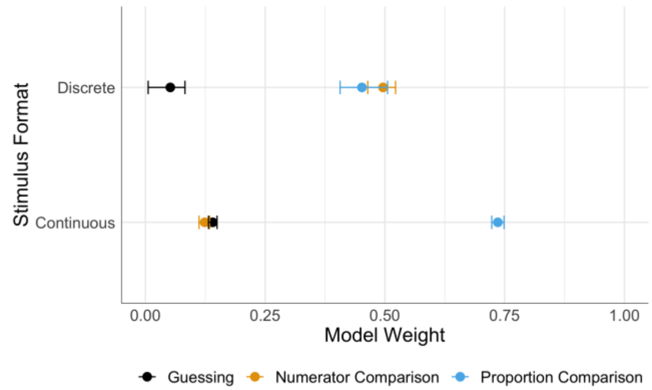

Figure 4: Mean mixture weights and 50% equal tailed intervals for each of the three strategy models when judging discrete and continuous stimuli.

With continuous stimuli, the model weights favored the proportion comparison model, .74 (.72, .75), over the numerator comparison model, .12 (.11, .13), or guessing, .14 (.13, .15). In contrast, with discrete stimuli the two comparison strategies were similarly weighted, with only a slight preference for the numerator comparison model, .50 (.46, .52), over the proportion comparison model, .45 (.41, .51), and with the guessing model again the least preferred, .05 (.01, .08).

Both the numerator comparison model and the proportion comparison model included one parameter, the weber ratio w, determining the precision with which the values are discriminated. For the numerator comparison model, the mean parameter estimates were $w = 0.28$ for discrete stimuli and $w = 0.18$ for continuous stimuli. For the proportion comparison model, the mean parameter estimates were $w = 0.49$ for discrete stimuli and $w = 0.09$ for continuous stimuli.

Discussion

In the current study, adult participants judged which of two probability machines, which varied in the proportional distribution of the two possible outcomes, would be more likely to result in a specific outcome. Overall, adults were able to accurately compare the proportions, but also showed evidence of numerical interference, with lower performance when the numerator was inconsistent with the correct response versus when it was consistent, and ratio dependent responding, with higher accuracy as the ratio between proportions increased. Moreover, although the ratio effects did not significantly differ for discrete and continuous trials, there was significantly larger numerical interference for discrete stimuli than for continuous stimuli. This behavioral pattern replicates substantial prior research with children and adults showing numerical interference in discrete contexts (e.g., Boyer et al., 2008; Hurst et al., 2021; Hurst & Cordes, 2018). However, it also reveals a more general phenomenon of interference from absolute information based on continuous area, suggesting that adults' numerical interference is not caused by a strict use of counting strategies.

Instead, we tested whether adults' behavior is captured by a strategy that approximates and compares the numerator magnitudes, relative to a strategy that approximates and compares the proportions. Consistent with our hypothesis, we find that adults are more likely to use an approximate proportion strategy when judging continuous proportion, but do not show a strong preference for either strategy when judging discrete proportion (although, there was a slight preference for an approximate numerator comparison strategy). Thus, one possible explanation for increased numerical interference in discrete proportional judgements is because people switch to a strategy that relies on comparing the numerators directly, rather than the holistic proportions.

However, it is unlikely that adults are entirely switching strategies for discrete proportional reasoning, compared to continuous proportional reasoning. Instead, the ratio dependent responding and similar model weights for the two strategies on discrete proportional reasoning suggests that people are attending to the holistic proportion, at least some of the time. Further, the numerical interference and low (but non-zero) model weights on the numerator comparison strategy for continuous proportional reasoning suggests that people may be attending to the numerator, beyond the holistic proportion, at least some of the time.

Moreover, it is worth noting that the estimated weber ratio on the numerators was much higher than is typically found for adults, both for numerical comparisons in the discrete stimuli and area-based comparisons in the continuous stimuli. Adults' typical weber ratios, representing the precision of the magnitudes representations, are 0.1 for both number and area (e.g., Odic et al., 2013). Here, however, we estimated a weber ratio of 0.28 for the discrete comparisons and 0.18 for the continuous comparisons, which correspond to the weber ratios typically found in 5-year-olds, rather than adults (e.g., Halberda & Feigenson, 2008; Odic et al., 2013). Thus, it may be that an exclusively ANS-based numerator comparison does not fully capture the kind of comparison that adults are doing. Instead, for example, their estimates of the numerator value might be less accurate than the equivalent magnitudes outside of a proportional reasoning context or they might be dynamically changing their strategy based on features of the comparison (e.g., DeWolf & Vosniadou, 2015; Jeong et al., 2007). For example, when the proportion magnitudes are sufficiently difficult to compare, adults may revert to an incorrect heuristic of using the numerator, even for continuous proportions.

Another possibility is that people attend to both the holistic proportion and the numerical components simultaneously, but over-weight the numerator information in a way that biases people's proportional reasoning (Alonso-Díaz et al., 2018). However, our model of approximate proportion comparison did not capture noise or bias caused by the estimation of individual components and instead treated proportion holistically. There may be other models that better incorporate perceptual biases and noise in proportion representation. For example, behavioral judgements of proportion typically result in overestimation of proportions less than 0.5 and underestimation for values greater than 0.5, resulting in an s-shaped curve (e.g,. Hollands & Dyre, 2000; Spence, 1990; Varey et al., 1990). Thus, proportional reasoning might be better represented by a model that incorporates this kind of non-linear representation (e.g., Zhang & Maloney, 2012). For example, one possibility is that each of the individual components (i.e., the numerator and its complement) are represented as a Gaussian activation, centered on the magnitude value and with some noise for that component. When these components are then integrated, they result in the behavioral patterns seen in proportion judgements (Gouet et al., 2021).

Importantly, the approach used here is easily scalable and future work could use the same model-based approach to compare any hypothesized strategy, including different models for correctly processing proportional information and incorrect models of heuristic or error-prone processes, providing a flexible approach for incorporating and testing these, and other, hypotheses. In addition to different strategies, this approach can also compare across different proportional formats or different task instructions. For example, in the continuous spinner representations used here adults could rely exclusively on the angle of the red segment, a strategy that we did not include in our strategy discovery model and a property that does not generalize to other continuous representations (e.g., rectangles or blobs). Additionally, the discrete dots used here varied in size. However, this may have introduced other strategies, such as overweighting the larger dots. Finally, directing people to attend to ratio, proportion, or percentages may change the kind of strategy they use (e.g., Varey et al., 1990). Future work can vary these properties of proportional formats and task instructions to better understand the variability in people's use strategy use.

An important next question is how people's use of these different strategies change across contexts and over the course of development and schooling. Although prior work has investigated age related changes in numerical interference, this work is typically limited to conclusions at a group level as to whether an age group shows numerical interference, or not (e.g., Boyer et al., 2008; Hurst & Cordes, 2018). The current model-based approach however provides a way to quantify the use of each hypothesized strategy, allowing for more direct comparisons across development that can capture continuous change. For example, we may be able to capture when children begin to rely on different strategies and whether children show an abrupt change in strategy use or whether they gradually incorporate different strategies into their reasoning.

In conclusion, these findings provide important insight into the mechanisms underlying absolute interference in proportional reasoning and provide a novel Bayesian model-based approach for quantifying strategy use in a way that is easily scalable to additional strategies and comparisons across contexts and development.

Acknowledgments

Research reported in this publication was supported by the NICHD of the National Institutes of Health under award number K99HD104990 to MAH. The content is solely the responsibility of the authors and does not necessarily represent the official views of the National Institutes of Health. We also thank Susan Levine for support with data collection and feedback on a prior draft.

References

Alonso-Díaz, S., Piantadosi, S. T., Hayden, B. Y., & Cantlon, J. F. (2018). Intrinsic whole number bias in humans. *Journal of Experimental Psychology: Human Perception and Performance*, *44*(9), 1472–1481. https://doi.org/10.1037/xhp0000544

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance*, *33*(6), 1410–1419. https://doi.org/10.1037/0096-1523.33.6.1410

Boyer, T. W., & Levine, S. C. (2015). Prompting children to reason proportionally: Processing discrete units as continuous amounts. *Developmental Psychology*, *51*(5), 615–620. https://doi.org/10.1037/a0039010

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology*, *44*(5), 1478–1490. https://doi.org/10.1037/a0013110

Braithwaite, D. W., & Siegler, R. S. (2018). Developmental changes in the whole number bias. *Developmental Science*, *21*(2), e12541. https://doi.org/10.1111/desc.12541

Brannon, E. M., Lutz, D., & Cordes, S. (2006). The development of area discrimination and its implications for number representation in infancy. *Developmental Science*, *9*(6), F59–F64. https://doi.org/10.1111/j.1467-7687.2006.00530.x

Bryant, P., & Nunes, T. (2012). *Children's understanding of probability: A literature review (summary report).* Nuffield Foundation.

Cantlon, J. F., Platt, M. L., & Brannon, E. M. (2009). Beyond the number domain. *Trends in Cognitive Sciences*, *13*(2), 83–91. https://doi.org/10.1016/j.tics.2008.11.007

Cordes, S., & Brannon, E. M. (2008). The Difficulties of Representing Continuous Extent in Infancy: Using Number Is Just Easier. *Child Development*, *79*(2), 476–489. https://doi.org/10.1111/j.1467-8624.2007.01137.x

Dehaene, S. (1992). Varieties of numerical abilities. *Cognition*, *44*(1–2), 1–42. https://doi.org/10.1016/0010-0277(92)90049-N

Dehaene, S. (2003). The neural basis of the Weber–Fechner law: A logarithmic mental number line. *Trends in Cognitive Sciences*, *7*(4), 145–147. https://doi.org/10.1016/S1364-6613(03)00055-X

Dehaene, S. (2011). *The number sense: How the mind creates mathematics*. Oxford University Press.

Denison, S., Reed, C., & Xu, F. (2013). The emergence of probabilistic reasoning in very young infants: Evidence from 4.5- and 6-month-olds. *Developmental Psychology*, *49*(2), 243–249. https://doi.org/10.1037/a0028278

Denison, S., & Xu, F. (2010). Twelve- to 14-month-old infants can predict single-event probability with large set sizes: Twelve- to 14-month-olds can predict single-event probability. *Developmental Science*, *13*(5), 798–803. https://doi.org/10.1111/j.1467-7687.2009.00943.x

Denison, S., & Xu, F. (2012). Probabilistic Inference in Human Infants. In *Advances in Child Development and Behavior* (Vol. 43, pp. 27–58). Elsevier. https://doi.org/10.1016/B978-0-12-397919-3.00002-2

DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance*, *40*(1), 71–82. https://doi.org/10.1037/a0032916

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction*, *37*, 39–49. https://doi.org/10.1016/j.learninstruc.2014.07.002

Fabbri, S., Caviola, S., Tang, J., Zorzi, M., & Butterworth, B. (2012). The role of numerosity in processing nonsymbolic proportions. *The Quarterly Journal of Experimental Psychology*, *65*(12), 2435–2446. https://doi.org/10.1080/17470218.2012.694896

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(1), 1.

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition*, *44*(1–2), 43–74.

Gelman, A., & Hill, J. (2007). *Data analysis using regression and multilevel/hierarchical models*. Cambridge University Press.

Girotto, V., Fontanari, L., Gonzalez, M., Vallortigara, G., & Blaye, A. (2016). Young children do not succeed in choice tasks that imply evaluating chances. *Cognition*, *152*, 32–39. https://doi.org/10.1016/j.cognition.2016.03.010

Gouet, C., Jin, W., Naiman, D. Q., Peña, M., & Halberda, J. (2021). Bias and noise in proportion estimation: A mixture psychophysical model. *Cognition*, 104805. https://doi.org/10.1016/j.cognition.2021.104805

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology*, *44*(5), 1457–1465. https://doi.org/10.1037/a0012682

Hoffman, M. D., & Gelman, A. (2014). The No-U-Turn Sampler: Adaptively Setting Path Lengths in Hamiltonian Monte Carlo. *Journal of Machine Learning Research*, *15*, 31.

Hollands, J. G., & Dyre, B. P. (2000). Bias in proportion judgments: The cyclical power model. *Psychological Review*, *107*(3), 500–524. https://doi.org/10.1037//0033-295X.107.3.500

Hurst, M. A., Boyer, T. W., & Cordes, S. (2021). Spontaneous and directed attention to number and proportion. *Journal of Experimental Psychology: Learning, Memory, and Cognition*. https://doi.org/10.1037/xlm0001084

Hurst, M. A., & Cordes, S. (2016). Rational-number comparison across notation: Fractions, decimals, and whole numbers. *Journal of Experimental Psychology: Human Perception and Performance*, *42*(2), 281–293. https://doi.org/10.1037/xhp0000140

Hurst, M. A., & Cordes, S. (2018). Attending to relations: Proportional reasoning in 3- to 6-year-old children. *Developmental Psychology*, *54*(3), 428–439. https://doi.org/10.1037/dev0000440

Hurst, M. A., & Cordes, S. (2019). Talking about proportion: Fraction labels impact numerical interference in non-symbolic proportional reasoning. *Developmental Science*, e12790. https://doi.org/10.1111/desc.12790

Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The Development of Proportional Reasoning: Effect of Continuous Versus Discrete Quantities. *Journal of Cognition and Development*, *8*(2), 237–256. https://doi.org/10.1080/15248370701202471

Kalra, P. B., Binzak, J. V., Matthews, P. G., & Hubbard, E. M. (2020). Symbolic fractions elicit an analog magnitude representation in school-age children. *Journal of Experimental Child Psychology*, *195*, 104844. https://doi.org/10.1016/j.jecp.2020.104844

Kushnir, T., Xu, F., & Wellman, H. M. (2010). Young Children Use Statistical Sampling to Infer the Preferences of Other People. *Psychological Science*, *21*(8), 1134–1140. https://doi.org/10.1177/0956797610376652

Meck, W. H., & Church, R. M. (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behavior Processes*, *9*(3), 320.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52.

Odic, D. (2018). Children's intuitive sense of number develops independently of their perception of area, density, length, and time. *Developmental Science*, *21*(2), e12533. https://doi.org/10.1111/desc.12533

Odic, D., Libertus, M. E., Feigenson, L., & Halberda, J. (2013). Developmental change in the acuity of approximate number and area representations. *Developmental Psychology*, *49*(6), 1103–1112. https://doi.org/10.1037/a0029472

Park, Y., Viegut, A. A., & Matthews, P. G. (2020). More than the sum of its parts: Exploring the development of ratio magnitude versus simple magnitude perception. *Developmental Science*. https://doi.org/10.1111/desc.13043

Piantadosi, S. T. (2016). Efficient estimation of Weber's W. *Behavior Research Methods*, *48*(1), 42–52. https://doi.org/10.3758/s13428-014-0558-8

Placì, S., Fischer, J., & Rakoczy, H. (2020). Do infants and preschoolers quantify probabilities based on proportions? *Royal Society Open Science*, *7*, 191751. https://doi.org/10.1098/rsos.191751

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology. Human Perception and Performance*, *36*(5), 1227.

Spence, I. (1990). Visual psychophysics of simple graphical elements. *Journal of Experimental Psychology: Human Perception and Performance*, *16*(4), 683–692. https://doi.org/10.1037/0096-1523.16.4.683

Stan Development Team. (2020). *RStan: The R interface to Stan* (R pacakge version 2.21.2) [Computer software]. http://mc-stan.org/

Téglás, E., Ibanez-Lillo, A., Costa, A., & Bonatti, L. L. (2015). Numerical representations and intuitions of probabilities at 12 months. *Developmental Science*, *18*(2), 183–193. https://doi.org/10.1111/desc.12196

Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science*, *185*(4157), 1124–1131.

Varey, C. A., Metters, B. A., & Birnbaum, M. H. (1990). Judgments of Proportions. *Journal of Experimental Psychology: Human Perception and Performance*, *16*(3), 613–625.

Walsh, V. (2003). A theory of magnitude: Common cortical metrics of time, space and quantity. *Trends in Cognitive Sciences*, *7*(11), 483–488. https://doi.org/10.1016/j.tics.2003.09.002

Xu, F., & Denison, S. (2009). Statistical inference and sensitivity to sampling in 11-month-old infants. *Cognition*, *112*(1), 97–104. https://doi.org/10.1016/j.cognition.2009.04.006

Xu, F., & Garcia, V. (2008). Intuitive statistics by 8-month-old infants. *Proceedings of the National Academy of Sciences*, *105*(13), 5012–5015. https://doi.org/10.1073/pnas.0704450105

Zhang, H., & Maloney, L. T. (2012). Ubiquitous Log Odds: A Common Representation of Probability and Frequency Distortion in Perception, Action, and Cognition. *Frontiers in Neuroscience*, *6*. https://doi.org/10.3389/fnins.2012.00001