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SYMBOLIC DYNAMICS ON FREE GROUPS

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Abstract. We study nearest-neighbor shifts of finite type (NNSOFT) on a free group $\mathcal{G}$. We determine when a NNSOFT on $\mathcal{G}$ admits a periodic coloring and give an example of a NNSOFT that does not allow a periodic coloring. Then, we find an expression for the entropy of the golden mean shift on $\mathcal{G}$. In doing so, we study a new generalization of Fibonacci numbers and analyze their asymptotics with a one-dimensional iterated map that is related to generalized continued fractions.

1. Introduction.

Let $\mathcal{G}$ be a free group that is freely generated by $\sigma = \{\sigma_1, \ldots, \sigma_q\}$ and fix a finite alphabet of colors $\mathcal{A}$. A *coloring* of $\mathcal{G}$ is a function $\phi : \mathcal{G} \to \mathcal{A}$. We will consider colorings to be points in the dynamical system $(\mathcal{A}^{\mathcal{G}}, \mathcal{G})$, where the action is given by

$$(w\phi)(x) = \phi(xw)$$

for $w, x \in \mathcal{G}$. Thus, $w\phi$ is a coloring that corresponds to "shifting" the coloring ϕ by the group element w. A coloring ϕ is *periodic* if the orbit $\{T\phi : T \in \mathcal{G}\}$ is finite.

A set $F \subset \mathcal{A} \times \sigma \times \mathcal{A}$ is called a set of *forbidden blocks*. A coloring ϕ *contains* $(a, \sigma_i, b) \in F$ if there exists $w \in \mathcal{G}$ such that $\phi(w) = a$ and $\phi(w\sigma_i) = b$. Thus F only restricts what colors can be adjacent over each of the σ_i. Given a set F of forbidden blocks, define X_F to be the set of colorings which do not contain any blocks in F. Since the cardinality of F is finite, we call X_F a *nearest-neighbor shift of finite type (NNSOFT)*.

Example 1. Let $\mathcal{A} = \{0, 1\}$, $\sigma = \{\sigma_1, \sigma_2\}$, and $F = \{\ (1, \sigma_i, 1) : \sigma_i \in \sigma\ \}$. Then X_F consists of all colorings of $\mathcal{G}$ such that no two 1s are adjacent. X_F is called the *golden mean shift* on $\mathcal{G}$.

We denote the length of a group element w as a reduced word on the generators of $\mathcal{G}$ by $|w|$.

Definition 1.1. The *entropy* $h(\mathsf{X}_F)$ of a NNSOFT X_F is defined as

$$h(\mathsf{X}_F) = \limsup_{n\to\infty} \frac{\log_2 B_n}{|C_n|}$$

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where $C_n = \{w \in \mathcal{G} : |w| \leq n\}$ and B_n is the number of allowed colorings of C_n in X_F.

Here, we first study NNSOFT on $\mathcal{G}$ and show that their behavior is considerably simpler than $\mathbb{Z}^d$ for $d > 1$. We show how to determine if a NNSOFT on $\mathcal{G}$ is nonempty (Proposition 1). This is in contrast to, for example, $\mathbb{Z}\times\mathbb{Z}$ where nonemptyness is not necessarily computable for NNSOFT [1]. We also determine when a NNSOFT on $\mathcal{G}$ admits a periodic coloring (Theorem 3.4), which can be reduced to finding a nontrivial nonnegative solution to a system of linear equations (Theorem 3.6). As in $\mathbb{Z}^d$ for $d > 1$, there exist NNSOFT on $\mathcal{G}$ that do not admit periodic colorings [1, 6], and an example is given in Section 4. In $\mathbb{Z} \times \mathbb{Z}$, the existence of NNSOFT that do not admit periodic coloring is closely related to the undecidability of $\mathbb{Z} \times \mathbb{Z}$ nonemptyness. On $\mathcal{G}$, we find no such relationship since there exist NNSOFT that do not allow periodic points, but nonemptyness is relatively easy to determine.

We also study the entropy of the golden mean shift on $\mathcal{G}$. In one dimension ($q = 1$), the entropy of the golden mean shift is easy to determine [8]. However, in $\mathbb{Z} \times \mathbb{Z}$ even determining if the entropy of the golden mean shift is algebraic is still an open problem. Much effort has been put towards achieving bounds on its value, which is important for understanding several physical systems and for many coding problems that arise in the study of two-dimensional run-length constrained channels [7, 9, 11]. The golden mean shifts on $\mathbb{Z}^d$ and $\mathcal{G}$ are instances of the *hard core* model from statistical mechanics, which corresponds to colorings of an arbitrary graph with the alphabet $\mathcal{A} = \{0, 1\}$ such that no two 1s are adjacent. Here we develop a new generalization of Fibonacci numbers to study the golden mean shift on $\mathcal{G}$ and find an expression for the entropy of the golden mean shift on $\mathcal{G}$ (Theorem 5.9). The results also hold for any k-regular trees, and thus extends to the hard core model on the Bethe lattice [2]. The unique simple invariant Gibbs measure on the Bethe lattice has been studied previously [10, 3].

2. **Coloring $\mathcal{G}$.** Fix a set of forbidden blocks F.

Definition 2.1. Let $\Gamma_{\sigma_i} = \{(a, b) \in \mathcal{A} \times \mathcal{A} : (a, \sigma_i, b) \notin F\}$ be the directed graph of allowed blocks of length 2 in the σ_i direction.

Every finite set of forbidden blocks F defines q one-dimensional shifts of finite type:

Definition 2.2. Let X_{σ_i} be the one-dimensional NNSOFT whose elements are given by bi-infinite walks on Γ_{σ_i}.

Informally, X_{σ_i} describes the space of possible colorings in the σ_i direction at each element of $\mathcal{G}$. It should be clear that the set of q one-dimensional shifts of finite type $X_{\sigma_1}, X_{\sigma_2}, \ldots, X_{\sigma_q}$ completely characterize X_F.

Example 2. Let $\mathcal{A} = \{0, 1\}$, $\sigma = \{\sigma_1, \sigma_2\}$, and $F = \{\ (1, \sigma_1, 1)\ \}$. Then X_{σ_1} is the golden mean shift and X_{σ_2} is the full shift on $\{0, 1\}$. Thus if $\phi \in \mathsf{X}_F$ and $\phi(w) = 1$, then $\phi(\sigma_1 w) = 0$, but it is not forbidden to have $\phi(\sigma_2 w) = 1$.

Definition 2.3. For any subset $\mathcal{A}' \in \mathcal{A}$, a set of $2q$ functions,

$$\{h_x : \mathcal{A}' \to \mathcal{A}' \text{ such that } x \in \sigma \text{ or } x^{-1} \in \sigma\}$$

is called a set of *coloring functions*.

Proposition 1. *X_F is nonempty if and only if for some $\mathcal{A}' \subseteq \mathcal{A}$, a set of coloring functions exist that obey the forbidden blocks, meaning that we always have $(l, \sigma_i, h_{\sigma_i}(l)) \notin F$ and $(h_{\sigma_i^{-1}}(l), \sigma_i, l) \notin F$ for every $l \in \mathcal{A}'$.*

Proof. It should be clear that if X_F is nonempty, then there must exist a set of coloring functions that obey the forbidden blocks since each element of $\mathcal{A}'$ can be adjacent to some other other element of $\mathcal{A}'$ by each σ_i, where $\mathcal{A}'$ is the set of colors used in some $\phi \in \mathsf{X}_F$.

For the converse direction, note that coloring functions can specify colorings in the following way. We begin by defining $\phi(e) = a$ for some $a \in \mathcal{A}'$. We then repeat the following process: if $\phi(w)$ has been defined but $\phi(wx)$ has not yet been defined for $x \in \sigma$ or $x^{-1} \in \sigma$, define $\phi(wx) = h_x(\phi(w))$. Therefore, once we have colored one vertex, the h_x inductively define the rest of the coloring. □

Note that since the domain and range of coloring functions are finite, one can trivially check if a NNSOFT is nonempty by simply enumerating all possible sets of coloring functions and determining if any are consistent with the forbidden blocks. It should be clear that if there exists an alphabet $\mathcal{A}' \subseteq \mathcal{A}$ such that for each $c \in \mathcal{A}'$ every X_{σ_i} contains a point that uses c and only the colors in $\mathcal{A}'$, then it is possible to construct a set of coloring functions and the NNSOFT will be nonempty.

In the proof of Theorem 3.4, we require the following result which states that the h_x are invertible, then the coloring specified by them will be periodic:

Proposition 2. *Let ϕ be a coloring produced by a set of coloring functions S. If $h_x^{-1} = h_{x^{-1}}$ for each $h_x \in S$, then ϕ is periodic.*

Proof. Note that a coloring ϕ specified by S is *color-isotropic*, meaning that if $\phi(g_1) = \phi(g_2)$ then $\phi(g_1 w) = \phi(g_2 w)$ for all $w \in \mathcal{G}$, as is evident from the fact that each h_x is a bijection. But every color-isotropic coloring is necessarily periodic, since any shift $T\phi$ of ϕ will color e one of a finite number of colors, and each color of e will uniquely determine a coloring of $\mathcal{G}$. □

3. **Periodic coloring of $\mathcal{G}$.** We now determine when a NNSOFT on $\mathcal{G}$ admits a periodic coloring. This question turns out to be much more subtle than it may initially seem. First, we represent a periodic point in a one-dimensional NNSOFT with a *cycle*:

Definition 3.1. A *cycle* w is a closed path on the graph Γ_{σ_i}. We represent a cycle through the vertices $x_1, x_2, \ldots x_n, x_1$ with $x_i \in \mathcal{A}$ by the expression $\overline{x_1 x_2 \ldots x_n}$.

A cycle $w = \overline{x_1 x_2 \ldots x_n}$ has *length* $|w| = n$ and we will sometimes write $(w)_i = x_i$. We say that w *represents a periodic point* $x \in X_{\sigma_i}$ if

$$x = \ldots x_1 x_2 \ldots x_n x_1 x_2 \ldots x_n . x_1 x_2 \ldots x_n \ldots .$$

Definition 3.2. A *simple cycle* is cycle $w = \overline{x_1 x_2 \ldots x_n}$ such that $x_i \neq x_j$ for $i \neq j$.

Thus, a simple cycle is a closed path on Γ_{σ_i} that does not contain any shorter closed paths.

Definition 3.3. For a cycle w and $a \in \mathcal{A}$, define

$$\eta_a(w) = \#\{i : (w)_i = a, 1 \leq i \leq |w|\}.$$

If S is a set of cycles, define

$$\eta_a(S) = \sum_{w \in S} \eta_a(w).$$

Theorem 3.4. *A NNSOFT on $\mathcal{G}$ on q generators contains a periodic coloring if and only if there exist finite sets $S_1, S_2, \ldots, S_q$ such that elements of S_i are cycles that represent periodic points in X_{σ_i}, and for all $a \in \mathcal{A}$ and $1 \le i < j \le q$ we have $\eta_a(S_i) = \eta_a(S_j)$.*

Proof. We first prove the $\Leftarrow$ direction. Suppose such S_i exist, and suppose $S_i = \{w_1^i, w_2^i, \ldots, w_{p_i}^i\}$ for each $1 \le i \le q$. Define the alphabet $\mathcal{B}$ by

$$\mathcal{B} = \bigcup_{1 \le i \le q} \bigcup_{j=1}^{|S_i|} \bigcup_{s=1}^{|w_j^i|} \{x_{j,s}^i\},$$

so that $\mathcal{B}$ consists of symbols such as $x_{1,1}^1$, $x_{1,2}^1$, $x_{2,1}^2$, etc.. Each element $x_{j,s}^i$ in $\mathcal{B}$ can be thought of as specifying a cycle w_j^i in S_i, and a position s in that cycle.

Next, we define $\chi : \mathcal{B} \to \mathcal{A}$ by

$$\chi(x_{j,s}^i) = (w_j^i)_s$$

so that

$$w_j^i = \overline{\chi(x_{j,1}^i)\chi(x_{j,2}^i) \ldots \chi(x_{j,|w_j^i|}^i)}.$$

That is, χ maps the symbol $x_{j,s}^i$ to the color at the location (in a cycle) specified by $x_{j,s}^i$.

Since we assume $\eta_a(S_i) = \eta_a(S_j)$ for all i, j and for all $a \in A$, we define η_a to be the common value of all $\eta_a(S_i)$. Define $N = \sum_{a \in \mathcal{A}} \eta_a$ and note we also have $N = \sum_{w \in S_i} |w|$ for every S_i.

Lemma 3.5. *Given $S_1, S_2, \ldots, S_q$ that satisfy the hypotheses of the theorem, there exist sets $E_1, E_2, \ldots E_N$ that partition $\mathcal{B}$ and meet the following conditions:*

(*i*) *If $x_{j,s}^i \in E_r$ and $x_{j',s'}^{i'} \in E_r$ then $\chi(x_{j,s}^i) = \chi(x_{j',s'}^{i'})$.*

(*ii*) *For $i = 1, 2, \ldots, q$, each E_r contains exactly one element of the form $x_{j,s}^i$ (where j and s may depend on i).*

Proof. In this proof we temporarily change notation. Suppose $\mathcal{A} = \{1, 2, 3, \ldots, m\}$ and

$$\mathcal{B} = \{b_1^1, b_2^1, \ldots, b_{p_1}^1, b_1^2, b_2^2, \ldots, b_{p_2}^2, \ldots b_1^q, b_2^q, \ldots, b_{p_q}^q\}$$

for some values of the p_r, where $b_x^i = x_{j,s}^i$ for some j, s. For each fixed $i \le q$, we can "sort" the b_x^i by color so that

$$\overbrace{b_{y_1^i}^i b_{y_2^i}^i \ldots b_{y_{\eta_1}^i}^i}^{\chi \text{ maps these to } 1 \in \mathcal{A}} \quad \overbrace{b_{y_{(\eta_1+1)}^i}^i \, b_{y_{(\eta_1+2)}^i}^i \ldots b_{y_{(\eta_1+\eta_2)}^i}^i}^{\chi \text{ maps these to } 2 \in \mathcal{A}} \ldots\ldots \overbrace{b_{y_{(N-\eta_m+1)}^i}^i \, b_{y_{(N-\eta_m+2)}^i}^i \ldots b_{y_{(N)}^i}^i}^{\chi \text{ maps these to } m \in \mathcal{A}},$$

for appropriate choices of the y_z^i. The conditions on the S_i assure that

$$\#\{b_x^i \in \mathcal{B} : \chi(b_x^i) = a\} = \#\{b_x^j \in \mathcal{B} : \chi(b_x^j) = a\}$$

for all $a \in \mathcal{A}$ and $1 \le i, j \le q$. Thus, when we sort by color, each row of b_x^i has the same number of elements (η_a) that χ maps to $a \in A$. That is, we can write

$$\begin{array}{cccc}
\overbrace{\begin{array}{c} b^1_{y^1_1} b^1_{y^1_2} \dots b^1_{y^1_{\eta_1}} \\ b^2_{y^2_1} b^2_{y^2_2} \dots b^2_{y^2_{\eta_1}} \\ \vdots \\ \vdots \\ b^q_{y^q_1} b^q_{y^q_2} \dots b^q_{y^q_{\eta_1}} \end{array}}^{\chi \text{ maps these to } 1 \in \mathcal{A}} &
\overbrace{\begin{array}{c} b^1_{y^1_{(\eta_1+1)}} \dots b^1_{y^1_{(\eta_1+\eta_2)}} \\ b^2_{y^2_{(\eta_1+1)}} \dots b^2_{y^2_{(\eta_1+\eta_2)}} \\ \vdots \\ \vdots \\ b^q_{y^q_{(\eta_1+1)}} \dots b^q_{y^q_{(\eta_1+\eta_2)}} \end{array}}^{\chi \text{ maps these to } 2 \in \mathcal{A}} &
\begin{array}{c} \dots \\ \dots \\ \vdots \\ \vdots \\ \dots \end{array} &
\overbrace{\begin{array}{c} b^1_{y^1_{(N-\eta_m+1)}} \dots b^1_{y^1_{(N)}} \\ b^2_{y^2_{(N-\eta_m+1)}} \dots b^2_{y^2_{(N)}} \\ \vdots \\ \vdots \\ b^q_{y^q_{(N-\eta_m+1)}} \dots b^q_{y^q_{(N)}} \end{array}}^{\chi \text{ maps these to } m \in \mathcal{A}}
\end{array}$$

for appropriate choices of y_a^i. To get the E_r, simply read down the column of this array. That is, let

$$E_r = \bigcup_{i=1}^{q} \{b^i_{y^i_r}\}.$$

Then the fact that the rows are sorted by color and each row contains the same number of each color implies (i). The fact that each E_y contains exactly one element from each row implies (ii). □

Continuing the proof of the main theorem, we can apply Lemma 3.5 to choose $E_1, E_2, \dots E_N$ that satisfy (i) and (ii). Note that (i) assures that elements of a given E_r all specify locations in cycles that are colored the same color, while (ii) assures that colors from cycles in each S_i are represented exactly once in each E_r, so that $|E_r| = |\sigma| = q$ for all r.

We will now use the E_r to construct a coloring of $\mathcal{G}$ with the alphabet $\{1, 2, \dots N\}$. Later, this will be projected down to a periodic coloring by $\mathcal{A}$. Given an $r \in \{1, 2 \dots, N\}$ and i with $1 \le i \le q$, find the unique $j, s \in \mathbb{Z}$ such that $x^i_{j,s} \in E_r$. Define $h_{\sigma_i}(r) = r'$, where r' is such that

$$x^i_{j,(s+1 \bmod |w^i_j|)} \in E_{r'}.$$

That is, given r, we find the element of the form $x^{\sigma_i}_{j,s} \in E_r$, and set $h_{\sigma_i}(x^{\sigma_i}_{j,s}) = r'$ such that $E_{r'}$ contains the symbol adjacent to $x^{\sigma_i}_{j,s}$ in the cycle $w^{\sigma_i}_j$. Each h_{σ_i} is invertible, so we can define $h_{\sigma_i^{-1}} = h^{-1}_{\sigma_i}$.

Note that each h_{σ_i} is necessarily a bijection, and therefore the h_{σ_i} define a periodic coloring by Proposition 2. Suppose this periodic coloring is $\Phi : \mathcal{G} \to \{1, 2, \dots, N\}$. We can project Φ down to a coloring of $\mathcal{G}$ by using χ. Define $\phi : \mathcal{G} \to \mathcal{A}$ by $\phi(w) = \chi(x)$, where x is any arbitrary element of $E_{\Phi(w)}$. Note the specific choice of x does not matter because of condition (i) above. The orbit $\{T\Phi : T \in \mathcal{G}\}$ is finite since Φ is periodic, and χ maps each of these colorings to at most one distinct coloring, so the orbit

$$\{T\phi : T \in \mathcal{G}\} = \chi(\{T\Phi : T \in \mathcal{G}\})$$

must be finite. This shows that ϕ is periodic.

To prove the $\Rightarrow$ direction, we show that any periodic coloring defines a set of cycles that meet the conditions of the theorem. Let ϕ be a periodic coloring that satisfies the NNSOFT constraints. We can consider the *stabilizer of* ϕ, $H = \{T \in \mathcal{G} : T\phi = \phi\}$. H is a finite index subgroup since ϕ is periodic, and can be thought of as a fundamental region of the periodic coloring.

H is also a normal subgroup, so consider the group of right cosets $\mathcal{G} \setminus H$. Since H is a stabilizer of ϕ, $h\phi(r) = \phi(r)$ for all $h \in H$ and $r \in \mathcal{G}$. Thus, elements of the coset (Ha) are all colored the color $\phi(a)$, but these colors may not be distinct for different cosets.

Now fix $\sigma_i \in \sigma$, and consider any coset (Hs). We define the *coset orbit*

$$O_{\sigma_i}(Hs) = \{(H\sigma_i)^n(Hs) : n \in \mathbb{Z}\}.$$

Since $\mathcal{G} \setminus H$ is finite, each coset orbit $O_{\sigma_i}(Hs)$ must be finite. Note that two coset orbits $O_{\sigma_i}(Hs_1)$ and $O_{\sigma_i}(Hs_2)$ are either equal or disjoint. Therefore, there are some finite number of distinct coset orbits,

$$O_{\sigma_i}(Hs_1),\, O_{\sigma_i}(Hs_2),\, \dots\, , O_{\sigma_i}(Hs_{m_i}),$$

which for each $\sigma_i \in \sigma$ form a disjoint partition of $\mathcal{G} \setminus H$. We can regard each coset orbit $O_{\sigma_i}(Hs_j)$ as specifying a cycle $C_{\sigma_i}(Hs_j)$ given by

$$C_{\sigma_i}(Hs_j) = \overline{\phi(s_j)\phi(s_j\sigma_i^1)\phi(s_j\sigma_i^2)\dots\phi(s_j\sigma_i^n)},$$

where n is the cardinality of $O_{\sigma_i}(Hs_j)$. Since H is the stabilizer of a coloring in X_F, each $C_{\sigma_i}(Hs_j)$ represents a point in X_{σ_i}.

We then define the set

$$S_{\sigma_i} = \{C_{\sigma_i}(Hs_j) : 1 \le j \le m_i\},$$

where m_i is the number of distinct coset orbits. Note that

$$\begin{aligned}
\eta_a(S_{\sigma_i}) =& \sum_{j=1}^{m_i} \eta_a(C_{\sigma_i}(Hs_j)) \\
=& \sum_{j=1}^{m_i} \#\{i : (C_{\sigma_i}(Hs_j))_i = a, 1 \le i \le |C_{\sigma_i}(Hs_j)|\} \\
=& \sum_{j=1}^{m_i} \#\{x : x \in O_{\sigma_i}(Hs_j), \chi(x) = a\} \\
=& \#\{(Hs) \in \mathcal{G} \setminus H : f(Hs) = a\},
\end{aligned}$$

since the $O_{\sigma_i}(Hs_j)$ form a disjoint partition of $\mathcal{G} \setminus H$. The value

$$\#\{(Hs) \in \mathcal{G} \setminus H : f(Hs) = a\}$$

is independent of σ_i, which shows that $\eta_a(S_i) = \eta_a(S_j)$ for all $\sigma_i, \sigma_j \in \sigma$. This shows that the S_i meet the hypotheses of the theorem.

□

Corollary 1. *If there is a one-dimensional periodic point x which is an element of X_{σ_i} for all $\sigma_i \in \sigma$, then X_F contains a periodic coloring.*

Proof. The hypotheses of the theorem are satisfied by choosing $S_i = \{y\}$ for all $i \le q$, where y is some cycle representing x. □

Theorem 3.6. *The existence of sets $S_1, S_2, \dots, S_q$ satisfying the conditions of Theorem 3.4 is equivalent to the existence of a nontrivial nonnegative solution to a system of linear equations.*

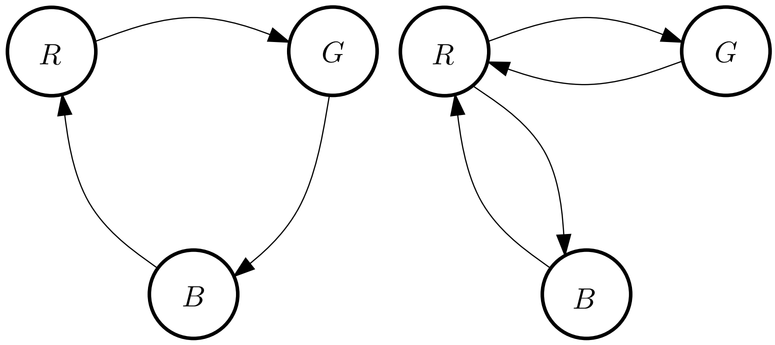

FIGURE 1. Shifts X_{σ_1} and X_{σ_2}.

Proof. Note that there are a finite number of simple cycles in each Γ_{σ_i} and let $c_{i,1}, c_{i,2}, \ldots, c_{i,l_i}$ be the set of simple cycles in Γ_{σ_i}. The fact that the $c_{i,r}$ are simple implies that $\eta_a(c_{i,r}) \in \{0,1\}$ for every $a \in \mathcal{A}$. Define $x_{i,r}$ to be the number of times that $c_{i,r}$ appears in all of the cycles of S_i. Then we have

$$\eta_a(S_i) = \sum_{r=1}^{l_i} x_{i,r}\eta_a(c_{i,r}).$$

Thus, the requirement of Theorem 3.4 that $\eta_a(S_i) = \eta_a(S_j)$ for $1 \leq i < j \leq q$ gives $q(q-1)/2$ homogeneous equations in the l_i unknowns, $x_{i,r}$, for each $a \in \mathcal{A}$. Since we must solve the system for all $a \in \mathcal{A}$, determining if a periodic coloring is allowed corresponds to solving $|\mathcal{A}|q(q-1)/2$ equations in $l_1 + l_2 + \ldots + l_q$ unknowns.

The entire system is given by a matrix with rational entries, so the existence of a nonnegative solution implies the existence of a nonnegative integer solution. □

4. **An example that is not periodic.** Now we present an example of a NNSOFT on $\mathcal{G}$ where each one-dimensional NNSOFT is irreducible, but $\mathcal{G}$ still cannot be colored periodically. This is perhaps a counterintuitive result because irreducibility is a strong condition and implies the existence of many kinds of periodic points.

Example 3. Consider the shifts of finite type shown in Figure 1 on the alphabet $\mathcal{A} = \{R, G, B\}$. Cycles representing points in X_{σ_1} are always of the form

$$\overline{RGBRGB\ldots RGB}.$$

Cycles representing points in X_{σ_2} are always of the form

$$\overline{Ra_1Ra_2\ldots Ra_{n-1}Ra_n},$$

where each $a_j \in \{\mathcal{G}, B\}$. This shows that for any set S_1 of cycles in X_{σ_1}, we must have

$$\eta_R(S_1) = \eta_G(S_1) = \eta_B(S_1). \tag{1}$$

In addition, for any set S_2 of cycles in X_{σ_2} we must have

$$\eta_R(S_2) = \eta_G(S_2) + \eta_B(S_2). \tag{2}$$

If $\eta_a(S_{\sigma_1}) = \eta_a(S_{\sigma_2}) = \eta_a$ for all $a \in \mathcal{A}$, then (1) and (2) become

$$\eta_R = \eta_G = \eta_B \tag{3}$$

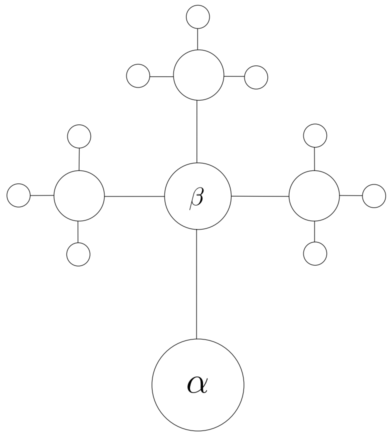

FIGURE 2. The tree T_3 for $k = 3$. Here group elements are represented as circles.

and

$$\eta_R = \eta_G + \eta_B. \tag{4}$$

Clearly we cannot simultaneously solve (3) and (4) unless $\eta_G = 0$ or $\eta_B = 0$, which would imply that S_1 is empty, since every point in X_{σ_1} uses G and B. Therefore, even though both X_{σ_1} and X_{σ_2} are irreducible, X_F does not allow a periodic coloring.

5. **Entropy of the golden mean shift on $\mathcal{G}$.** Let X_F denote the golden mean shift on the free group $\mathcal{G}$ on q free generators, defined in Example 1 of the introduction. For notational simplicity we define $k = 2q - 1$. However, the results of this section are general enough to apply to any integer $k > 1$.

We first analyze finite blocks in X_F by using a combinatorial argument. Fix any $\sigma_i \in \sigma$ and define the set $T_n \subset \mathcal{G}$ by

$$T_n = \{e\} \cup \{\sigma_i w : w \in \mathcal{G}, |w| \leq n-1\}$$

for $n \geq 1$. T_n is called a *tree* and can be represented by a subset of the Cayley graph of $\mathcal{G}$. Such a graph is shown in Figure 2. In the graph of T_n, we label the vertex corresponding to e by α and the vertex corresponding to σ_i by β. We call α the "root" of T_n and we say that T_n has *height* n.

Definition 5.1. Let $c_0(n)$ and $c_1(n)$ be the numbers of possible colorings of T_n when α is colored 0 and 1, respectively.

Theorem 5.2. *$c_0(n)$ satisfies the following recursion relationship: $c_0(1) = 2$, $c_0(2) = 2^k + 1$, and $c_0(n) = [c_0(n-1)]^k + [c_0(n-2)]^{k^2}$ for all $n \geq 2$.*

Proof. It is easy to see that $c_0(1) = 2$ and $c_0(2) = 2^k + 1$.

When α is colored 1, then β must be colored 0. But we can regard β as the "root" of k different trees each of height $n-1$. The colorings of each of the k trees for which β is the "root" can be chosen independently. Thus,

$$c_1(n) = [c_0(n-1)]^k. \tag{5}$$

Similarly, when α is colored 0, β can be colored either 0 or 1, so

$$c_0(n) = [c_0(n-1)]^k + [c_1(n-1)]^k ,$$

from which the result follows. $\square$

Define $C_n = \{w \in \mathcal{G} : |w| \leq n\}$ and denote the number of colorings of C_n allowed in X_F by B_n.

Theorem 5.3. *B_n satisfies $B_n = [c_0(n)]^{k+1} + [c_0(n-1)]^{k(k+1)}$.*

Proof. Note that C_n consists of $k+1$ trees of height n that all share e as their root. When e is colored 0, there are $c_0(n)$ possible colorings of each of the $k+1$ trees of height n, each of which can be chosen independently, giving a total of $[c_0(n)]^{k+1}$ possible colorings.

When e is colored 1, each of the $k+1$ trees has a root colored 1, and each tree can be colored independently. Therefore, there are $[c_1(n)]^{k+1} = [c_0(n-1)]^{k(k+1)}$ possible colorings of C_n when e is colored 1. $\square$

For notational simplicity, define $a_n = c_0(n)$ so that we have the recursion formula

$$a_{n+1} = a_n^k + a_{n-1}^{k^2} \tag{6}$$

with $a_1 = 2$ and $a_2 = 2^k + 1$.When $k = 1$, so that the free group under consideration is $\mathbb{Z}$, the a_n are Fibonacci numbers with the standard recursion formula. When $k \neq 1$, the nonlinear recursion sequence a_n will be central for understanding the golden mean shift on the free group.

We begin by studying the general growth properties of (a_n). Define $q_n = a_n/a_{n-1}^k$. We first determine when

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}^k} = \lim_{n\to\infty} q_n$$

exists. We will show that the limit exists for sufficiently small k and equals the solution of $x^{k+1} = x^k + 1$ in the interval $I = [1, 2]$. When k is sufficiently large, the limit may not exist but the values of q_n oscillate between two limits. In this case, the odd terms and even terms respectively converge to different limits.

Proposition 3. *For each $k = 2, 3, \ldots$ we have that $q_2 < q_3$ and $q_2 < q_4$.*

Proof. For the first part, $q_2 < q_3$ only if $a_2^{k+1} < a_1^k a_3$. Now,

$$a_2^{k+1} = (2^k + 1)^{k+1} = (2^k + 1)^k (2^k + 1) = 2^k (2^k + 1)^k + (2^k + 1)^k,$$

and

$$a_1^k a_3 = a_1^k (a_2^k + a_1^{k^2}) = 2^k (2^k + 1)^k + 2^{k^2+k}.$$

This shows $a_2^{k+1} < a_1^k a_3$ only if $(2^k+1)^k < 2^{k^2+k}$. But $(2^k+1)^k < (2^{k+1})^k = 2^{k^2+k}$, so we must have $q_2 < q_3$.

Note $q_2 < q_4$ only if

$$a_2 a_3^k < a_4 a_1^k.$$

We have that

$$a_2 a_3^k = (2^k + 1) a_3^k = 2^k a_3^k + a_3^k$$

and

$$a_4 a_1^k = (a_3^k + a_2^{k^2}) 2^k = 2^k a_3^k + 2^k a_2^{k^2},$$

so $q_2 < q_4$ only if $a_3^k < 2^k a_2^{k^2}$. This is the same as requiring that

$$a_3 < 2 a_2^k,$$

which is the same as

$$a_2^k + a_1^{k^2} < 2a_2^k.$$

This simplifies to $a_1^{k^2} < a_2^k$, or equivalently $2^{k^2} < (2^k+1)^k$, which is obviously true. □

Proposition 4. *For each $n = 2, 3, \dots$ we have*

$$(i)\ q_{n+1} < q_{n-1} \quad \textit{implies } q_n < q_{n+2}$$
$$(ii)\ q_{n+1} > q_{n-1} \quad \textit{implies } q_n > q_{n+2}.$$

Proof. We will prove only (i), as the proof for (ii) is analogous. Note that $q_{n+1} < q_{n-1}$ if and only if

$$a_{n-2}^{k^2} a_{n+1}^k < a_{n-1}^k a_n^{k^2}.$$

Adding $a_{n-1}^k a_{n+1}^k$ to both sides and factoring gives

$$(a_{n-1}^k + a_{n-2}^{k^2}) a_{n+1}^k < a_{n-1}^k (a_{n+1}^k + a_n^{k^2}),$$

which is equivalent to

$$a_n a_{n+1}^k < a_{n-1}^k a_{n+2}.$$

This is the same as $q_{n+2} > q_n$. □

Let $E_n = q_{2n}$ and $O_n = q_{2n+1}$ for $n = 1, 2, \dots$ denote the subsequences of (q_n) which consist of the even- and odd-numbered terms of (q_n) respectively. Equation (6) implies $q_{n+1} = 1 + 1/q_k^k$. Therefore, we can understand the sequence (q_n) by studying the iterated map $f : [1,2] \to [1,2]$ given by $f(x) = 1 + 1/x^k$. Define α to be the unique fixed point of f in $[1,2]$ and note that α satisfies $\alpha^{k+1} = \alpha^k + 1$.

Proposition 5. *(E_n) converges monotonically to a limit $L_E \in [1,\alpha]$ and (O_n) converges monotonically to a limit $L_O \in [\alpha, 2]$.*

Proof. This follows from Proposition 3, Proposition 4, and the fact that the q_n are bounded. □

To simplify notation, we also define a function $g : [1,2] \to [1,2]$ by $g(x) = f^2(x)$. Note that $g(q_n) = q_{n+2}$, so that $g(O_n) = O_{n+1}$ and $g(E_n) = E_{n+1}$. This implies that the (possibly non-distinct) fixed points of g are L_E, L_O, and α. It is not difficult to show that $x = \sqrt[k]{k-1}$ is a maximum for $g'(x)$ on $[1,2]$ and that $g'(\sqrt[k]{k-1}) < 1$ only when

$$(k-1)^{k+1} < k^k.$$

Definition 5.4. Let $\Omega = \inf \{ k : (k-1)^{k+1} < k^k \} \approx 4.14104$.

Note that for for $k > \Omega$ we have $(k-1)^{k+1} > k^k$ and for $k < \Omega$ we have $(k-1)^{k+1} < k^k$. Thus, for $k < \Omega$ we know that $\sup \{ g'(x) : x \in [1,2] \} = C < 1$ for all x. Therefore, for $k < \Omega$ the Contraction Mapping Theorem implies that g has a unique fixed point in $[1,2]$. This implies that the sequences (E_n) and (O_n) both converge to $L_E = L_O = \alpha$. Thus, for $k < \Omega$, we know that the limit

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{a_n}{a_{n-1}^k} = \alpha,$$

where α is the root of

$$\alpha^{k+1} - \alpha^k - 1 = 0 \tag{7}$$

in $I = [1,2]$.

Next, we show that for $k > \Omega$ there are two distinct fixed points besides α, one strictly less than α and one strictly greater than α. Therefore, $L_E \neq L_O$, and neither equals α.

Lemma 5.5. *$k > \Omega$ implies $g'(\alpha) > 1$.*

Proof. $g'(\alpha) = f'(f(\alpha))f'(\alpha) = [f'(\alpha)]^2 = k^2(\alpha^2)^{-k-1} > 1$ if and only if $k^2 > (\alpha^2)^{k+1}$, or, equivalently, $k > \alpha^{k+1}$. But $\alpha^{k+1} = \alpha^k + 1$, so $k > \alpha^{k+1}$ only if $\sqrt[k]{k-1} > \alpha$. When $k > \Omega$ we have that

$$f(\sqrt[k]{k-1}) = 1 + \frac{1}{k-1} = \frac{k}{k-1} < \sqrt[k]{k-1},$$

which implies $\sqrt[k]{k-1} > \alpha$, since f is decreasing and α is the unique fixed point of f. □

Theorem 5.6. *For $k > \Omega$, we have that α, L_E, and L_O are distinct and the only three values in $I = [1, 2]$ for which $g(x) = x$.*

Proof. We have shown that g must have fixed points L_O and L_E such that $E_n \uparrow L_E$ and $O_n \downarrow L_O$. For $k > \Omega$, Lemma 5.5 shows that α is a repelling fixed point, which implies that we cannot have $L_E = \alpha$ or $L_O = \alpha$, because L_E and L_O are the limit points of $g^n(q_2)$ and $g^n(q_3)$ respectively. In addition, $L_E \neq L_O$ since $L_E \in [1, \alpha]$ and $L_O \in [\alpha, 2]$.

To see that there cannot be more than three fixed points, note that $g'(1) < 1$ and $g'(2) < 1$. Since g'' has only one zero in I, this implies that $g'(x) = 1$ for at most two values in I. Therefore, g can cross the line $y = x$ at most three times. □

We now determine the asymptotics of (a_n), in which the limits L_E and L_O will arise naturally. We denote exponentiation with base 2 by $\exp_2(x)$. By the definition of (q_n) we have,

$$a_n = 2^{k^{n-1}} \prod_{j=2}^{n} q_j^{k^{n-j}} = \exp_2 \left[k^{n-1} \left(1 + k \sum_{j=2}^{n} \frac{\log_2 q_j}{k^j} \right) \right].$$

Define the sum

$$A_n = 1 + k \sum_{j=2}^{n} \frac{\log_2 q_j}{k^j}, \tag{8}$$

so that $a_n = \exp_2(k^{n-1} A_n)$. Since the q_j are bounded, (A_n) must converge to some limit, A, given by

$$A = 1 + k \sum_{j=2}^{\infty} \frac{\log_2 q_j}{k^j}. \tag{9}$$

Note that A_n, A, and q_j all depend on k. It is reasonable to guess that a_n is asymptotic to $\lambda \exp_2(k^{n-1} A)$ for some λ, and this is indeed true.

Theorem 5.7. *For $k = 2, 3, \ldots$ and n even, we have $a_n \sim \lambda \exp_2(k^{n-1} A)$ with*

$$\lambda = \exp_2 \left(-\frac{\log_2 L_E + k \log_2 L_O}{k^2 - 1} \right)$$

where A is as above and L_O and L_E are the limits of (O_n) and (E_n) respectively.

Proof. We will prove this by showing that

$$\lim_{n\to\infty} \frac{\exp_2(k^{n-1}A)}{a_n} = \lim_{n\to\infty} \exp_2\left(k^{n-1}(A-A_n)\right) = \exp_2\left(\frac{\log_2 L_E + k\log_2 L_O}{k^2-1}\right).$$

We define

$$W_n = k^{n-1}(A-A_n) = \sum_{j=1}^{\infty} k^{-j}\log_2 q_{j+n}. \tag{10}$$

For all $r \geq 3$, we have $1 < q_r < 2$, so $\log_2 q_r > 0$. Then we can rearrange the sum in (10), splitting it into even and odd terms to give

$$W_n = \left[\sum_{j=1}^{\infty} k^{-2j}\log_2 q_{2j+n} + \sum_{j=1}^{\infty} k^{-2j+1}\log_2 q_{(2j-1)+n}\right]. \tag{11}$$

Fix $\epsilon > 0$. We have already shown that $(O_n) \to L_O$ and $(E_n) \to L_E$, so choose N such that for all $n > N$ we have

$$|\log_2 O_n - \log_2 L_O| < \epsilon \text{ and } |\log_2 E_n - \log_2 L_E| < \epsilon.$$

Then we know that for n even,

$$\begin{aligned}
&\left|W_n - \left((\log_2 L_E)\sum_{j=1}^{\infty} k^{-2j} + (\log_2 L_O)\sum_{j=1}^{\infty} k^{-2j+1}\right)\right| \\
=&\left|\sum_{j=1}^{\infty} k^{-2j}(\log_2 q_{2j+n} - \log_2 L_E) + \sum_{j=1}^{\infty} k^{-2j+1}(\log_2 q_{2j-1} - \log_2 L_O)\right| \qquad (12)\\
<&\epsilon\sum_{j=1}^{\infty} k^{-j} \leq \epsilon.
\end{aligned}$$

Therefore for n even,

$$W_n \to (\log_2 L_E)\sum_{j=1}^{\infty} k^{-2j} + (\log_2 L_O)\sum_{j=1}^{\infty} k^{-2j+1} = \frac{\log_2 L_E + k\log_2 L_O}{k^2-1}.$$

□

Theorem 5.8. *For $k = 2, 3, \ldots$ and n odd, we have $a_n \sim \lambda\exp_2(k^{n-1}A)$ with*

$$\lambda = \exp_2\left(-\frac{\log_2 L_O + k\log_2 L_E}{k^2-1}\right)$$

where A is as above and L_E and L_O are the limits of (O_n) and (E_n) respectively.

Proof. The proof is identical to the proof of the previous theorem, except that for odd n, L_E and L_O are switched in (12) and every step after. □

Note that for n even we have,

$$\lambda = \exp_2\left(-\frac{\log_2 L_E + k\log_2 L_O}{k^2-1}\right)$$

and for n odd we have,

$$\lambda = \exp_2\left(-\frac{\log_2 L_O + k\log_2 L_E}{k^2-1}\right).$$

When $k < \Omega$, we know that $L_E = L_O = \alpha$, so these expressions both simplify to

$$\lambda = \exp_2\left(-\frac{(k+1)\log_2\alpha}{k^2-1}\right) = \exp_2\left(-\frac{\log_2\alpha}{k-1}\right) = \alpha^{\frac{1}{1-k}}.$$

Theorem 5.9. *The entropy of the golden mean shift on the free group $\mathcal{G}$ is given by $h(\mathsf{X}_F) = A(k-1)/k$.*

Proof. To prove the theorem we show that if $a_n \sim \lambda \exp_2(Ak^{n-1})$, then

$$\lim_{n\to\infty} \frac{\log_2 B_n}{|C_n|} = A\frac{k-1}{k}.$$

In the free group,

$$|C_n| = 1 + \sum_{j=1}^{n}(k+1)k^{j-1} = 1 + (k+1)\frac{k^n-1}{k-1}.$$

Next by Theorem 5.3,

$$\lim_{n\to\infty} \frac{\log_2 B_n}{|C_n|} = \lim_{n\to\infty} \frac{\log_2\left[a_n^{k+1} + a_{n-1}^{k(k+1)}\right]}{|C_n|}.$$

Substituting for a_n and simplifying gives,

$$\lim_{n\to\infty} \frac{\log_2\left[\lambda^{k+1}\exp_2((k+1)Ak^{n-1})u_n + \lambda^{k(k+1)}\exp_2((k+1)Ak^{n-1})u_{n-1}\right]}{|C_n|},$$

where $u_n \to 1$. Factoring and simplifying gives,

$$\begin{aligned}\lim_{n\to\infty} \frac{\log_2 B_n}{|C_n|} &= \lim_{n\to\infty} \frac{(k+1)Ak^{n-1} + \log_2\left[\lambda^{k+1}u_n + \lambda^{k(k+1)}u_{n-1}\right]}{|C_n|} \\ &= \lim_{n\to\infty} \frac{(k+1)k^{n-1}A}{1 + (k+1)\frac{k^n-1}{k-1}} \\ &= A\frac{k-1}{k}.\end{aligned}$$

□

Remark 1. Since λ does not appear in the expression for $h(\mathsf{X}_F)$, it is irrelevant to the entropy calculation that λ may depend on whether we look at the even or the odd terms of a_n.

Remark 2. Iterations of the map $f(x) = 1 + x^{-k}$ produce an expression similar to the continued fraction for the golden mean, ϕ, but with k'th powers on each denominator. The above results show that for $k < \Omega$, the limit $f^n(x)$ as $n \to \infty$ is independent of x. Such continued fractions with exponents on the denominator are called *generalized continued fractions* and can be defined using a nonlinear version of the Gauss map given by $\phi_k(x) = x^{-1/k} \mod 1$. It is easy to show that for $k > \Omega$ generalized continued fractions are not well-defined. Note that ϕ_k acts as the shift on generalized continued fraction expansions and is conjugate to the usual Gauss map where $k = 1$. Generalized continued fractions and their invariant measures have been investigated in [4, 5].

Theorem 5.9 shows that the entropy of the golden mean shift on $\mathcal{G}$ is therefore given by

$$h(\mathsf{X}_F) = A\frac{k-1}{k} = \frac{k-1}{k}(1 + k\sum_{j=2}^{\infty} k^{-j}\log_2 q_j). \tag{13}$$

The form of (13) is useful for numerical approximation because the q_j are bounded, so an upper bound can easily be found. The below table shows upper and lower bounds on the entropy. Lower bounds were found by taking n terms of Equation (13) for each k. Upper bounds were found by taking n terms of the sum and then taking the $q_j = 2$ for $j = n + 1$ to ∞.

k	n	lower bound	upper bound
2	20	0.7341852591	$+9.5 \cdot 10^{-7}$
3	15	0.7748799619	$+7.0 \cdot 10^{-8}$
4	10	0.8096380469	$+9.5 \cdot 10^{-7}$
5	10	0.8372473361	$+1.0 \cdot 10^{-7}$
6	10	0.8587923404	$+1.7 \cdot 10^{-8}$

Finding a closed form for $h(\mathsf{X}_F)$ is still an open problem. We did find a closed form for λ when $k < \Omega$, which implies that perhaps it will be easier in other problems, such as the hard core model on $\mathbb{Z}\times\mathbb{Z}$, to find the constant in front of the exponential, rather than the actual entropy.

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REFERENCES

[1] R. Berger, *The undecidability of the domino problem*, Mem. Amer. Math. Soc., **66** (1966).
[2] H. A. Bethe, *Statistical theory of superlattices*, Proc. Roy. Soc. London Ser A, **150** (1935), 552-575.
[3] G. Brightwell and P. Winkler, *A second threshold for the hard-core model on a Bethe lattice*, Random Structures Algorithms, **24** (2004), 303–314.
[4] G. H. Choe, *Generalized continued fractions*, Applied Mathematics and Computation, **109** (2000), 287–299.
[5] G. H. Choe and C. Kim, *The Khintchine constants for generalized continued fractions*, Applied Mathematics and Computation, **144** (2003), 397–411.
[6] K. Culik II and J. Kari, *An aperiodic set of Wang cubes,* Journal of Universal Computer Science, **1** (1995), 675–686
[7] A. Kato and K. Zeger, *On the capacity of two-dimensional run-length constrained channels*, IEEE Trans. Inform. Theory, **45** (1999), 1527–1540.
[8] D. Lind and B. Marcus, "An Introduction to Symbolic Dynamics and Coding," Cambridge University Press, Cambridge, 1995.
[9] P. Siegel, R. Roth and J. Wolf, *Efficient coding schemes for the hard square model*, IEEE Trans. Inform. Theory, **47** (2001), 1166–1176.
[10] Y. Suhov and U. A. Rozikov, *A Hard-Core model on a Cayley Tree: An example of a loss network*, Queuing Systems, **46** (2004), 197–212.
[11] W. Weeks and R. Blahut, *The capacity and coding gain of certain checkerboard codes*, IEEE Trans. Inform. Theory, **44** (1998), 1193–1203.

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